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HYERS-ULAM STABILITY OF QUADRATIC FUNCTIONAL EQUATIONS

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Abstract

In this paper,we establish the general solution and the generalized Hyers-Ulam stability problem for the equation $f(2x+y)+f(2x-y)=f(x+y)+f(x-y)+6f(x),$ (1)

1. Introduction

In 1940, S.M.Ulam [20] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of homomorphisms:

It is significant for us to decrease the possible estimator of the stability problem for the functional equations. This work is possible if we consider the stability problem in the of Hyers-Ulam-Rassias for a functional equations(1). As a reselt, we have much better possible upper bounds for the equations (1) than those of Czerwik [4] and Skof-Cholewa[3].

Solution of $f(2x+y)+f(2x-y)=f(x+y)+f(x-y)+6f(x),$

Let $\mathbb{R}^{+}$ denote the set of all nonnegative real numbers and let both E_1 and E_2 be the vector spaces.

We here present the general solution of (1)

Theorem 1

Let $\phi : X^2 \to \mathbb{R}^{+}$ be a function such that

$$\sum_{i=0}^{\infty}\frac{\phi(2^i x,0)}{4^i} \qquad \left(\sum_{i=1}^{\infty}4^i\phi\left(\frac{x}{2^i},0\right), respectively\right) \tag{2}$$

Converges and

$$\lim_{n\to\infty}\frac{\phi(2^n x,2^n y)}{4^n}=0 \qquad \left(\lim_{n\to\infty}4^n\phi\left(\frac{x}{2^n},\frac{y}{2^n}\right)=0\right), \forall x,y\in E_{1.} \tag{3}$$

Suppose that a function $f:X\to Y$ Satisfies

$$\| f(2x+y)+f(2x-y)-f(x+y)-f(x-y)-6f(x)\|\le\phi(x,y), \forall x,y\in E_{1.} \tag{4}$$

For all $x,y\in X$. Then there exists a unique quadratic function $T:X\to Y$ Which Satisfies the equation (2.3) and the inequality

$$\| f(x)-T(x)\|\le\frac{1}{8}\sum_{i=0}^{\infty}\frac{\phi(2^i x,0)}{4^i} \tag{5}$$

$$\left(\| f(x)-T(x)\|\le\frac{1}{8}\sum_{i=1}^{\infty}4^i\phi\left(\frac{x}{2^i},0\right)\right),$$

for all $x\in X$. The function T is given by

$$T(x)=\lim_{n\to\infty}\frac{f(2^n x)}{4^n} \qquad \left(T(x)=\lim_{n\to\infty}4^n f\left(\frac{x}{2^n}\right)\right) \tag{6}$$

for all $x\in X$.

Proof:

Putting y = 0 in $f(2x+y)+f(2x-y)=f(x+y)+f(x-y)+6f(x)$, and divided by 8, we have

$$\left\|\frac{f(2x)}{4}-f(x)\right\|\le\frac{1}{8}\phi(x,0) \tag{7}$$

for all $x\in X$. Replacing x by 2x in (7) and dividing by 4 and summing the resulting inequality with (7), we get

$$\left\|\frac{f(2^2x)}{4^2}-f(x)\right\|\le\frac{1}{8}\left[\phi(x,0)+\frac{\phi(2x,0)}{4}\right] \tag{8}$$

for all $x\in X$. Using the induction on a positive integer n,we obtain that

$$\left\|\frac{f(2^n x)}{4^n}-f(x)\right\|\le\frac{1}{8}\sum_{i=0}^{n-1}\frac{\phi(2^i x,0)}{4^i} \tag{9}$$

$$\le\frac{1}{8}\sum_{i=0}^{\infty}\frac{\phi(2^i x,0)}{4^i}$$

for all $x\in X$. In order to prove convergence of the sequence $\left\{\frac{f(2^n x)}{4^n}\right\}$, we divide inequality(9) by 4^m and also replace x by $2^m x$ to find that for n,m>0,

$$\left\| \frac{f(2^n 2^m x)}{4^{n+m}} - \frac{f(2^m x)}{4^m} \right\| = \frac{1}{4^m} \left\| \frac{f(2^n 2^m x)}{4^n} - f(2^m x) \right\| \tag{10}$$

$$\le \frac{1}{8.4^m} \sum_{i=0}^{n-1} \frac{\phi(2^i 2^m x, 0)}{4^i}$$

$$\le \frac{1}{8} \sum_{i=0}^{\infty} \frac{\phi(2^i 2^m x, 0)}{4^{m+i}}.$$

Since the right hand side of the inequality tends to 0 as m tends to infinity,the sequence $\left\{ \frac{f(2^n x)}{4^n} \right\}$ is a Cauchy sequence. Therefore, we may define $T(x) = \lim_{n\to\infty} 2^{-2n} f(2^n x)$ for all $x \in X$.

By letting $n \to \infty$ in (9), we arrive at the formula (5).

To show that T satisfies the equation (2.3), replace x,y by $2^n x, 2^n y$,respectively in

$f(2x+y) + f(2x-y) = f(x+y) + f(x-y) + 6f(x)$, and divided by 4^n ,then it follows that

$$4^{-n} \left\| f(2^n(2x+y)) + f(2^n(2x-y)) - f(2^n(x+y)) - f(2^n(x-y)) - 6f(2^n x) \right\| \le 4^{-n} \phi(2^n x, 2^n y).$$

Taking the limits as $n \to \infty$,we find that T satisfies (2.3) for all $x, y \in X$.

To prove the uniqueness of the quadratic function T subject to (1), let us assume that there exists a quadratic function $S: X \to Y$ which satisfies (2.3) and the inequality (1).

Obviously,we have $S(2^n x) = 4^n S(x)$ and $T(2^n x) = 4^n T(x)$ For all $x \in X$ and $n \in \mathbb{N}$ .Hence it follows from (1) that $\| S(x) - T(x) \| = 4^{-n} \| S(2^n x) - T(2^n x) \|$

$$\le 4^{-n} \left(\left\| S(2^n x) - f(2^n x) \right\| + \left\| f(2^n x) - T(2^n x) \right\| \right)$$

$$\le \frac{1}{4} \sum_{i=0}^{\infty} \frac{\phi(2^i 2^n x, 0)}{4^{n+i}}$$

For all $x \in X$. By letting $n \to \infty$ in the preceding inequality,we immediately find the uniqueness of T. This completes the proof of the theorem.

Throughout this paper,Let B be a unital Banach algebra with norm|.|, and let ${}_B B_1$ and ${}_B B_2$ be the left Banach B-modules with norm||.|| and ||.||,respectively.

A quadratic mapping $Q: {}_B B_1 \to {}_B B_2$ is called B-quadratic if

$Q(ax) = a^2 Q(x), \quad \forall a \in B, \forall x \in {}_B B_{1.}$

Corollary1.1.

Let $\phi: {}_B B_1 \times {}_B B_1 \to \mathbb{R}^+$ be a function satisfies (1) and (2) for all $x, y \in {}_B B_{1.}$ Suppose that a mapping $f: {}_B B_1 \to {}_B B_2$ satisfies

$$\left\| f(2\alpha x + \alpha y) + f(2\alpha x - \alpha y) - \alpha^2 f(x+y) - \alpha^2 f(x-y) - 6\alpha^2 f(x) \right\| \le \phi(x, y)$$

For all $\alpha \in B(|\alpha| = 1)$ and for all $x, y \in {}_B B_{1,}$ and f is measurable or f(tx) is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B B_{1.}$ Then there exists a unique B-quadratic mapping $T: {}_B B_1 \to {}_B B_2$, defined by(5), which satisfies the equation (2.3) and the inequality (1) for all $x \in {}_B B_{1.}$

Proof:

By theorem 3.1, it follows from the inequality of the statement for $\alpha = 1$ that there exists a unique quadratic mapping $T : {}_B B_1 \to {}_B B_2$ satisfying the inequality(3.4) for all $x \in {}_B B_{1.}$ Under the assumption that f is measurable or f(tx) is continuous in $x \in \mathbb{R}$ for each fixed $x \in {}_B B_{1,}$ by the same reasoning as the proof of [5],The quadratic mapping $T : {}_B B_1 \to {}_B B_2$ satisfies $T(tx) = t^2 T(x)$, $\forall x \in {}_B B_{1,} \forall t \in \mathbb{R}$.

That is,T is $\mathbb{R}$ -quadratic. For each fixed $\alpha \in B(|\alpha| = 1)$, replacing f by T and setting y = 0 in (2.3),we have $T(\alpha x) = \alpha^2 T(x)$ for all $x \in {}_B B_{1,}$ The last relation is also true for $\alpha = 0.$ For each element $\alpha \in B(\alpha \neq 0), a = |a|.\frac{a}{|a|}.$

Since T is $\mathbb{R}$ -quadratic and $T(\alpha x) = \alpha^2 T(x)$ for each element $\alpha \in B(|\alpha| = 1)$,

$$T(ax) = T\left(|a|.\frac{a}{|a|}x\right)$$

$$= |a|^2 .T\left(\frac{a}{|a|}x\right)$$

$$= |a|^2 .\frac{a^2}{|a|^2}.T(x)$$

$$= a^2 T(x), \quad \forall a \in B(a \neq 0), \forall x \in {}_B B_{1.}$$

So the unique $\mathbb{R}$ -quadratic mapping $T : {}_B B_1 \to {}_B B_2$, is also B-quadratic,as desired.

This completes the proof of the corollary.

Corollary 1.2.

Let E_1 and E_2 be Banach spaces over the complex field $\mathbb{C}$, and let $\in \geq 0$ be a real number. Suppose that a mapping f: $E_1 \to E_2$ satisfies

$$\left\| f(2\alpha x + \alpha y) + f(2\alpha x - \alpha y) - \alpha^2 f(x+y) - \alpha^2 f(x-y) - 6\alpha^2 f(x) \right\| \leq \in$$

For all $\alpha \in \mathbb{C}(|\alpha| = 1)$ andfor all $x, y \in E_{1,}$ and f is measurable or $f(tx)$ continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Then there exists a unique $\mathbb{C}$ -quadratic mapping $T : E_1 \to E_2$ which satisfies the equation (1.3) and the inequality

$$\left\| f(x) - T(x) \right\| \leq \frac{\in}{6}, \forall x \in E_{1.}$$

Corollary 1.3

Let X and Y be a real normed space and Banach space,respectively, and let $\in, p, q$ be real numbers such that $\in \geq 0, q > 0$ and either $p, q < 2$ or $p, q > 2$. Suppose that a function $f : X \to Y$ satisfies

$$\left\| f(2x+y) + f(2x-y) - f(x+y) - f(x-y) - 6f(x) \right\| \leq \in \left(\|x\|^p + \|y\|^q \right)$$

for all $x, y \in X.$ Then there exists a unique quadratic function $T : X \to Y$ which satisfies the equation (1.3) and the inequality

$$\|f(x)-T(x)\| \le \frac{\in}{2|4-2^p|}\|x\|^p$$

for all $x \in X$ and for all $x \in X-\{0\}$ if p<0.

The function T is given by $T(x)=\lim_{n\to\infty}\frac{f(2^n x)}{4^n}$ if p,q<2 $\left(T(x)=\lim_{n\to\infty}4^n f\left(\frac{x}{2^n}\right) if p,q>2\right)$ for all $x \in X$. Further, if for each fixed $x \in X$ the mapping $t \to f(tx)$ from $\mathbb{R}$ to Y is continuous, then $T(rx)=r^2T(x)$ for all $r \in \mathbb{R}$.

The proof of the corollary.

Corollary 1.4

Let X and Y be a real normed space and a Banach space,respectively, and let $\in \ge 0$ be real number. Suppose that a function $f: X \to Y$ satisfies

$$\|f(2x+y)+f(2x-y)-f(x+y)-f(x-y)-6f(x)\| \le \in \tag{11}$$

For all $x, y \in X$. Then there exists a unique quadratic function $T: X \to Y$ defined by $T(x)=\lim_{n\to\infty}\frac{f(2^n x)}{4^n}$ which satisfies the equation (1.3) and the inequality

$$\|f(x)-T(x)\| \le \frac{\in}{6} \tag{12}$$

$x \in X$. Further, if for each fixed $x \in X$ the mapping $t \to f(tx)$ from $\mathbb{R}$ to Y is continuous, then $T(rx)=r^2T(x)$ for all $r \in \mathbb{R}$.

Corollary 1.5

Let X and Y be a real normed space and Banach space,respectively, and let $\in \ge 0, 0<p\ne 2$ be real number. Suppose that a function $f: X \to Y$ satisfies

$$\|f(2x+y)+f(2x-y)-f(x+y)-f(x-y)-6f(x)\| \le \in\left(\|x\|^p+\|y\|^p\right)$$

for all $x, y \in X$. Then there exists a unique quadratic function $T: X \to Y$ which satisfies the equation (1.3) and the inequality

$\|f(x)-T(x)\| \le \frac{5\in}{2|9-3^p|}\|x\|^p$ for all $x \in X$. The function T is given by

$T(x)=\lim_{n\to\infty}\frac{f(3^n x)}{9^n}$ if 0<p<2 $\left(T(x)=\lim_{n\to\infty}9^n f\left(\frac{x}{3^n}\right) if p>2\right)$

for all $x \in X$. Further, if for each fixed $x \in X$ the mapping $t \to f(tx)$ from $\mathbb{R}$ to Y is continuous, then $T(rx)=r^2T(x)$ for all

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